

Original article

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Dynamical control on the Adomian decomposition method for solving shallow water wave equation

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Abstract: The aim of this study is to apply a novel technique to control the accuracy and error of the Adomian decomposition method (ADM) for solving nonlinear shallow water wave equation. The ADM is among semi-analytical and powerful methods for solving many mathematical and engineering problems. We apply the Controle et Estimation Stochastique des Arrondis de Calculs (CESTAC) method which is based on stochastic arithmetic (SA). Also instead of applying mathematical packages we use the Control of Accuracy and Debugging for Numerical Applications (CADNA) library. In this library we will write all codes using C++ programming codes. Applying the method we can find the optimal numerical results, error and step of the ADM and they are the main novelties of this research. The numerical results show the accuracy and efficiency of the novel scheme.

Keywords: shallow water wave problem, Adomian decomposition method, CESTAC method, CADNA library

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Научная статья

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Численная валидация метода разложения Адомиана для решения нелинейного уравнения мелкой воды

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Аннотация. Целью данной работы является применение новой методики для проведения эффективного контроля точности и анализа погрешности применения метода разложения Адомиана для решения нелинейного волнового уравнения мелкой воды, возникающего при выполнении ряда важных задач в различных областях машиностроения и теории материалов. Рассмотрены три важных случая: набегание жидкости на полуплоские берега, на берега с умеренным уклоном и на берега с крутым уклоном. В исследованиях применялся метод раз-

ложения Адомиана, являющийся полуаналитическим эффективным методом, обладающим большей гибкостью, чем прямое разложение в ряд Тейлора. В данном исследовании мы применяем метод CESTAC (Contrôle et Estimation STochastique des Arrondis de Calculs), основанный на стохастической арифметике. Также вместо применения стандартных математических пакетов в работе эффективно использована библиотека контроля точности и отладки для численных приложений CADNA (Control of Accuracy and Debugging for Numerical Applications). Программная реализация подхода с использованием библиотеки CADNA выполнена на C++ под операционную систему LINUX. Вместо использования традиционной абсолютной ошибки, которая основана на точном решении и небольшом пороговом значении, используем новое правило останова, которое основано на двух последовательных приближениях. Основная теорема метода CESTAC показывает, что количество общих значащих цифр двух последовательных приближений практически равно количеству общих значащих цифр точного и приближенного решений. Применение представленной в работе методики позволило получить оптимальные численные результаты: найти погрешность и оптимальный шаг метода разложения Адомиана, чего не позволяли делать классические подходы. В этом заключается основная новизна работы. Приведены результаты тестирования разработанной численной модели для решения уравнения мелкой воды. Таким образом, при проведении численных расчетов с использованием предлагаемого метода разложения Адомиана продемонстрирована высокая точность и эффективность разработанного подхода для решения нелинейного волнового уравнения мелкой воды.

Ключевые слова: волновая задача на мелководье, метод разложения Адомиана, метод CESTAC, библиотека CADNA

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INTRODUCTION

Finding numerical and accurate methods for solving wave equation is one of important topics in applied mathematics [1–26]. The problem has special and important applications to simulate Tsunami phenomenon. Tsunamis can be categorized as long waves. In general form we apply the solitary waves or combinations of negative and positive solitary-like waves to simulate this phenomenon. Also many other applications of the shallow water wave problems can be found in the field of metallurgy and materials science. The horizontal centrifugal casting using shallow water model has been discussed by Boháček et al. in [4]. Also Bresch et al. in [5] have focused on some compressible fluid models specially the shallow water system. In [13] Nobel and Vila have discussed the thin power-law film flow

down an inclined plane: consistent shallow-water models.

In this study we discuss the following problem which is the specific case of the run-up of two dimensional long waves incident upon a uniform sloping beach connected to an open ocean with a uniform depth (fig. 1).

We consider the following nonlinear shallow-water equations:

$$\begin{aligned} \eta_t + (u(h + \eta))_x &= 0; \\ u_t + uu_x + g\eta_x &= 0, \end{aligned} \quad (1)$$

where wave domain is displayed by η , the depth averaged velocity is showed by u , the variable depth is presented by h , the acceleration of gravity is exhibited by g . Also the initial conditions are given

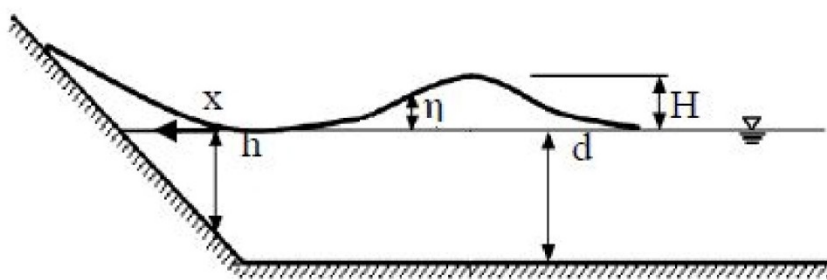


Fig. 1. Definition Sketch for solitary wave run-up
Рис. 1. Эскиз определения для набега уединенной волны

$$\begin{aligned}\eta(x, 0) &= H \operatorname{sech}^2 \sqrt{\frac{3H}{4d^3}} x; \\ u(x, 0) &= \frac{\eta}{d} \sqrt{gd},\end{aligned}\quad (2)$$

where the initial wave height and stationary elevation are displayed by H and d respectively.

Noeiaghdam et al. in [15] applied the homotopy analysis method for solving this problem. Approximate traveling wave solution has been discussed by some authors in [1, 12]. A new analytical solution for nonlinear shallow water-wave equation can be found in [3]. In [2] the stability analysis of two dimensional extended shallow water wave equation has been illustrated. Also in [22] the space-time fractional shallow water wave equation has been studied.

Adomian decomposition method is among semi-analytical and powerful methods for solving many kinds of linear and non-linear problems. Solving Volterra integral equation with discontinuous kernel [18], first kind integral equations with hyper-singular kernels [20], ordinary differential equations [10] and fuzzy Convection-Diffusion equation [11] are only some of applications of the ADM.

In the mentioned papers and many other studies the computations have been obtained using floating point arithmetic (FPA). In FPA in order to show the accuracy we should apply the traditional conditions which are based on the exact solution and ε as follows

$$|Q - Q_J| \leq \varepsilon, \text{ or } |Q_J - Q_{J-1}| \leq \varepsilon. \quad (3)$$

In real life problems that we do not have exact solution it will be impossible to use the conditions. Also the optimal value of ε is unknown for us. Thus choosing small or large values for ε either we will not be able to find the accurate results or we will have many iterations without improving the accuracy.

In this study instead of applying FPA, we use SA and a novel condition based on successive iterations Q_J and Q_{J-1} as

$$|Q_J - Q_{J-1}| = @.0. \quad (4)$$

@.0 shows the informatical zero which can be produced only in SA when we apply the CESTAC method and the CADNA library. It shows that number of common significant digits (NCSDs) of two successive iterations are almost zero. The CESTAC method has been presented by Vignes and Laporte [21]. Also a French research team has been improved the method and also the library [6–9]. In [16] the CESTAC method has been applied to control the accuracy of a nonlinear fractional order model of COVID-19. A dynamical control on the reverse osmosis system can be found in [14, 19]. Also in [17] the CESTAC method has been used to validate the results of numerical methods for solving integral equations.

In this paper, we apply the famous Adomian decomposition method for solving problem (1). Also using the CESTAC method and the CADNA library we try to control the accuracy of the results. The main novelty of this study is to find the optimal approximations, optimal error and optimal step of the method.

ADOMIAN DECOMPOSITION METHOD

Consider the following problem:

$$Lu + R(u) + F(u) = g(t), \quad (5)$$

where L is the operator of the highest-ordered derivatives with respect to t and R is the remainder of the linear operator. Also we show the nonlinear term with $F(u)$. We can write

$$Lu = g(t) - R(u) - F(u). \quad (6)$$

Now we can define the following inverse operator:

$$L_t^{-1} = \int_0^t (\cdot) dt. \quad (7)$$

Applying the inverse operator L_t^{-1} for both sides of (7) we get

$$u = f_0 + L_t^{-1}[g(t) - R(u) - F(u)], \quad (8)$$

where f_0 is the solution of homogeneous equation $Lu = 0$ involving the constants of integration. The integration constants involved in the

solution of the mentioned homogeneous equation are to be determined by the initial or boundary condition according as the problem is initial-value problem or boundary-value problem.

According to the ADM we can assume that the solution can be obtained in the form of the following series solution:

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \quad (9)$$

and also for the nonlinear term $F(u)$ we have

$$F(u) = \sum_{n=0}^{\infty} A_n, \quad (10)$$

$$\begin{cases} L_t \eta(x, t) + L_x[u(x, t)h(x)] + L_x[u(x, t)\eta(x, t)] = 0, \\ L_t u(x, t) + u(x, t)L_x u(x, t) + gL_x \eta(x, t) = 0. \end{cases} \quad (13)$$

Now we can apply the inverse operator L_t^{-1} for both sides of (13) as

$$\begin{cases} \eta(x, t) = \eta(x, 0) - L_t^{-1}L_x[u(x, t)h(x)] - L_t^{-1}L_x[u(x, t)\eta(x, t)], \\ u(x, t) = u(x, 0) - L_t^{-1}[u(x, t)L_x u(x, t)] - gL_t^{-1}L_x[\eta(x, t)]. \end{cases} \quad (14)$$

According to the traditional ADM for nonlinear terms $u\eta$ and uu_x the following terms can be applied

$$u\eta = \sum_{n=0}^{\infty} A_n, uu_x = \sum_{n=0}^{\infty} B_n, \quad (15)$$

where

$$A_0 = u_0\eta_0,$$

$$A_1 = u_0\eta_1 + u_1\eta_0,$$

$$A_2 = u_0\eta_2 + u_1\eta_1 + u_2\eta_0, \quad (16)$$

$$A_3 = u_0\eta_3 + u_1\eta_2 + u_2\eta_1 + u_3\eta_0,$$

⋮

and

$$\begin{cases} \sum_{n=0}^{\infty} \eta_n(x, t) = \eta(x, 0) - L_t^{-1}L_x[\sum_{n=0}^{\infty} u_n(x, t)h(x)] - L_t^{-1}L_x[\sum_{n=0}^{\infty} A_n], \\ \sum_{n=0}^{\infty} u_n(x, t) = u(x, 0) - L_t^{-1}[\sum_{n=0}^{\infty} B_n] - gL_t^{-1}L_x[\sum_{n=0}^{\infty} \eta_n(x, t)]. \end{cases} \quad (19)$$

where

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [F(\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, n = 0, 1, 2, \dots \quad (11)$$

MAIN IDEA

Applying the following operators:

$$L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, \quad (12)$$

and also the inverse operator $L_t^{-1} = \int_0^t (\cdot) dt$ on (1) we get

$$B_0 = u_{0x}u_0,$$

$$B_1 = u_{0x}u_1 + u_{1x}u_0,$$

$$B_2 = u_{0x}u_2 + u_{1x}u_1 + u_{2x}u_0, \quad (17)$$

$$B_3 = u_{0x}u_3 + u_{1x}u_2 + u_{2x}u_1 + u_{3x}u_0,$$

⋮

Thus substituting

$$\eta(x, t) = \sum_{n=0}^{\infty} \eta_n(x, t), \quad (18)$$

$$u(x, t) = \sum_{n=0}^{\infty} u_n(x, t),$$

in (14) we get

Following Adomian approach we obtain the following recursive relation:

$$\begin{aligned} \eta_0(x, t) &= \eta(x, 0), \\ u_0(x, t) &= u(x, 0), \\ &\vdots \\ \eta_{k+1}(x, t) &= -L_t^{-1}L_x[u_k(x, t)h(x)] - L_t^{-1}L_x[A_k], \\ u_{k+1}(x, t) &= -L_t^{-1}[B_k] - gL_t^{-1}L_x[\eta_k(x, t)]. \end{aligned} \tag{20}$$

The m -th order approximate solution can be obtained using the following relations

$$\begin{aligned} \eta_m(x, t) &= \sum_{n=0}^m \eta_n(x, t), \\ u_m(x, t) &= \sum_{n=0}^m u_n(x, t). \end{aligned} \tag{21}$$

CESTAC method-CADNA library. Assume that some representable values are produced by computer and they are collected in set A . Then $W \in A$ can be produced for $w \in R$ with \mathcal{R} mantissa bits of the binary FPA in the following form

$$W = w - \chi 2^{E-\mathcal{R}} \xi, \tag{22}$$

$$C_{\chi_1, \chi_2} = \begin{cases} \log_{10} \left| \frac{\chi_1 + \chi_2}{2(\chi_1 - \chi_2)} \right| = \log_{10} \left| \frac{\chi_1}{\chi_1 - \chi_2} - \frac{1}{2} \right|, & \chi_1 \neq \chi_2, \\ +\infty, & \chi_1 = \chi_2. \end{cases} \tag{23}$$

Theorem 1. Applying the ADM for solving the shallow water wave problem (1), the NCSDs of the exact and approximate solutions are almost equal to the NCSDs of two successive approximations as

$$\begin{aligned} C_{\eta_m, \eta_{m+1}} &\approx C_{\eta, \eta_m}; \\ C_{u_m, u_{m+1}} &\approx C_{u, u_m}. \end{aligned}$$

NUMERICAL ILLUSTRATIONS

Assuming $H = 2$ and $d = 20$, we discuss the results in three forms 1 – Semi-flat shores ($h(x) = 0.2x - 20$), 2 – Moderate-slope shores ($h(x) = x - 100$), 3 – Sharp-slope shores

where sign of w showed by χ , missing segment of the mantissa presented by $2^{-\mathcal{R}} \xi$ and the binary exponent of the result characterized by E . Moreover, in single and double precisions $\mathcal{R} = 24,53$ respectively [6, 21, 23].

Assume ξ is the casual variable that uniformly distributed on $[-1,1]$. After making perturbation on final mantissa bit of w we will have (μ) and (σ) as mean and standard deviation for results of W which they have important role in accuracy of W . Repeating this process J times for $W_i, i = 1, \dots, J$ we will have quasi Gaussian distribution for results. It means that μ for these data equals to the exact w . It is clear that we should find μ and σ based on $W_i, i = 1, \dots, J$. For more consideration, the following algorithm is presented where τ_δ is the value of T distribution as the confidence interval is $1 - \delta$ with $J - 1$ freedom degree [7–9]. We should note that in the CESTAC method instead of applying usual applications such as Mathematica and Maple we use the CADNA library [23].

Definition 1. [15, 16] Number of common significant digits for two real numbers χ_1, χ_2 can be defined as

($h(x) = 5x - 500$). We apply the ADM for solving problem (1) and we use the CESTAC method and the CADNA library to validate the results. Fig. 2, 3 and 4 show the approximate solutions for the mentioned shores for $x = t = 0.1$. According to results of tab. 1 and 2, optimal approximation and optimal error of method for solving shallow water wave equation are obtained. Thus the optimal approximation of $u(x, t)$ is 2.0006415328545538, the optimal step is $m_{opt} = 6$ and the optimal error 4.5×10^{-5} . Also, for the approximate solution of $\eta(x, t)$ is 1.401153672165483, the optimal step is $m_{opt} = 6$ and the optimal error 2.8×10^{-5} .

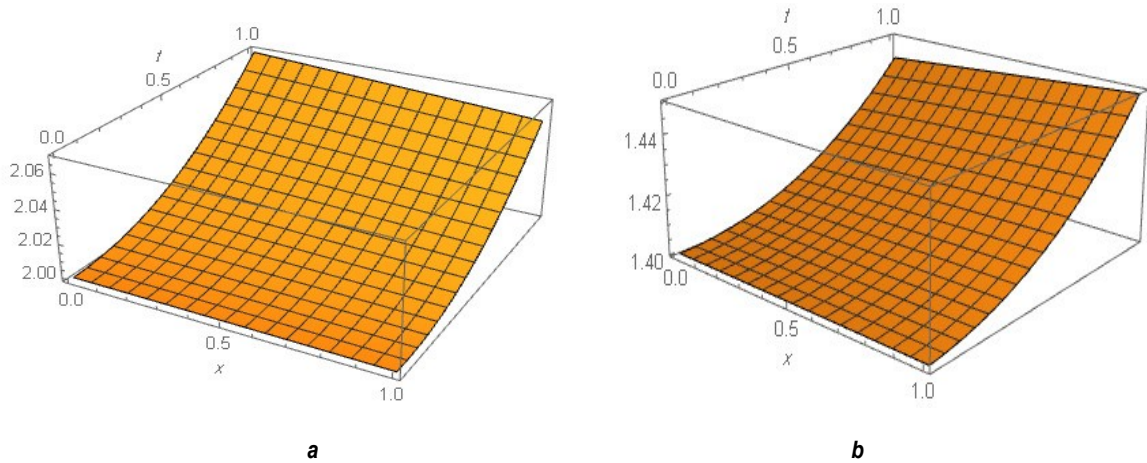


Fig. 2. Approximate solution for semi-flat shores mode: a – $\eta(x,t)$; b – $u(x,t)$
Рис. 2. Приближенное решение для режима пологих берегов: а – $\eta(x,t)$; б – $u(x,t)$

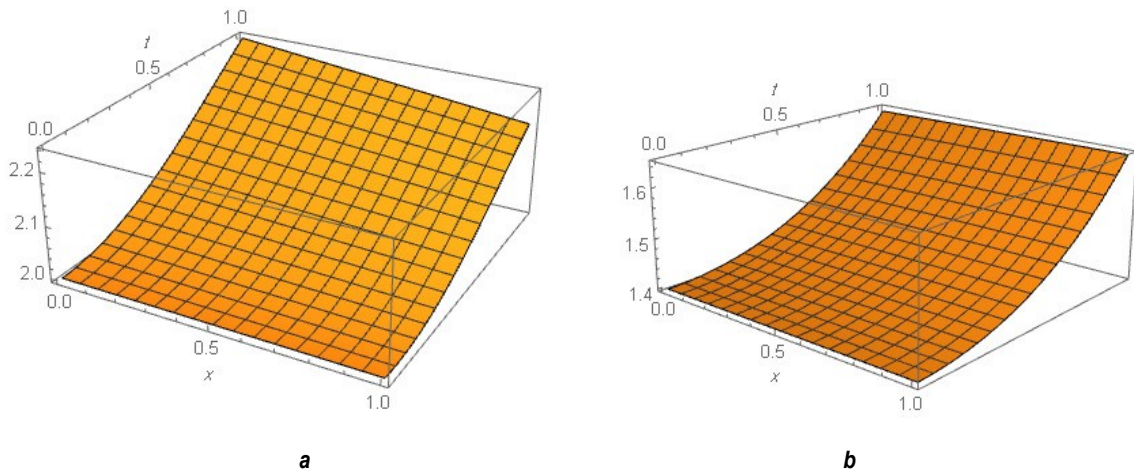


Fig. 3. Approximate solution for moderate-slope shores mode: a – $\eta(x,t)$; b – $u(x,t)$
Рис. 3. Приближенное решение для режима берегов с умеренным уклоном: а – $\eta(x,t)$; б – $u(x,t)$

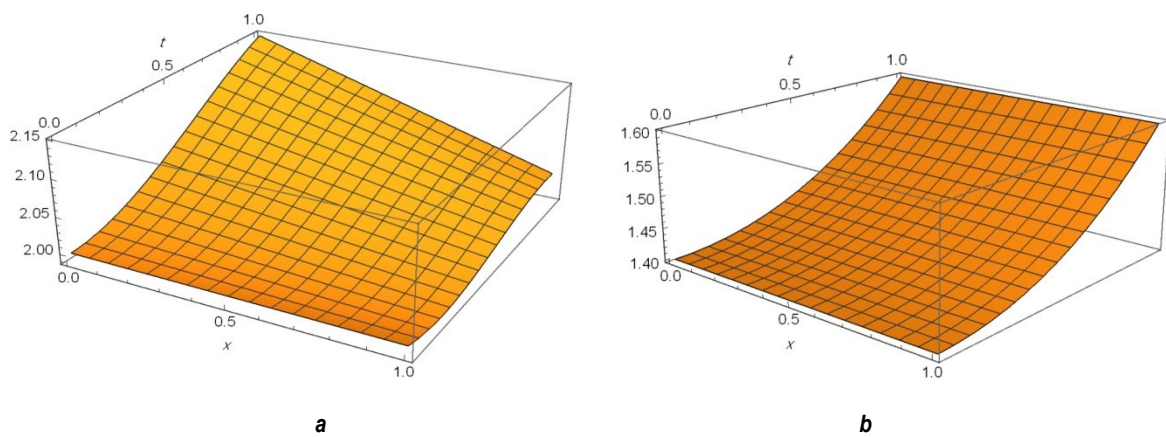


Fig. 4. Approximate solution for sharp-slope shores mode: a – $\eta(x,t)$; b – $u(x,t)$
Рис. 4. Приближенное решение для режима берегов с резким уклоном: а – $\eta(x,t)$; б – $u(x,t)$

Table 1. Applying the CESTAC method for numerical results of $u(x, t)$ **Таблица 1.** Применение метода CESTAC для численных результатов $u(x, t)$

m	u_m	$ u_m - u_{m+1} $
1	1.9999962462046976	1.999996246
2	1.9999489773636574	4.72688×10^{-5}
3	2.000322299202164	3.7325×10^{-4}
4	2.000504545245128	1.8231×10^{-4}
5	2.0005981472797174	9.36×10^{-5}
6	2.0006434267503708	4.5×10^{-5}
7	2.0006415328545538	@.0.

Table 2. Applying the CESTAC method for numerical results of $\eta(x, t)$ **Таблица 2.** Применение метода CESTAC для численных результатов $\eta(x, t)$

m	η_m	$ \eta_m - \eta_{m+1} $
1	1.4006973710294601	1.4007
2	1.4007378405649158	4.04695×10^{-5}
3	1.4009610914243738	2.2325×10^{-4}
4	1.4010754534197702	1.143×10^{-4}
5	1.401131280443496	5.58×10^{-5}
6	1.401159855883387	2.8×10^{-5}
7	1.401153672165483	@.0.

CONCLUSIONS

The ADM is a good and accurate method for solving linear and non-linear problems. Applying the method we found the approximate solution of the non-linear shallow water wave problem. Combining the ADM with the CESTAC method

we tried to find the optimal approximation, step and error of the method. We applied the CADNA library to find the results. Plotting the graphs of approximate solution for semi-flat, moderate-slope and sharp-slope shores we showed the efficiency of the method.

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