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# A new criterion of asymptotic stability for Hopfield neural networks with time-varying delay

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***Abstract.*** The objective of this paper is to analyze the stability of Hopfield neural networks with time-varying delay. For the system to operate in a steady state, it is important to guarantee the stability of Hopfield neural networks with time-varying delay. The Lyapunov-Krasovsky functional method is the main method for investigating the stability of time-delayed systems. On the basis of this method, the stability of Hopfield neural networks with time-varying delay is analysed. It is known that due to such factors as communication time, limited switching speed of various active devices, time delays often arise in various technical systems, which significantly degrade the performance of the system, which can in turn lead to a complete loss of stability. In this regard, a Lyapunov-Krasovsky type delay-product functional was constructed in the paper, which allows more information about the time delay and reduces the conservatism of the method. Then a generalized integral inequality based on the free matrix was used. A new criterion for asymptotic stability of Hopfield neural networks with time-varying delay, which has less conservatism, was formulated. The effectiveness of the proposed method is illustrated. Thus an asymptotic stability criterion for Hopfield neural networks with time-varying delay was formulated and justified. The expanded Lyapunov-Krasovsky functional is constructed on the basis of delay and quadratic multiplicative functional, and the derivative of the functional is defined by a matrix integral inequality with free weights. The effectiveness of the method is illustrated by a model example.
***Keywords:*** hopfield neural networks, asymptotical stability, LKF method, time-varying delay
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ЭНЕРГЕТИКА

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# Новый критерий асимптотической устойчивости нейронных сетей Хопфилда с переменным запаздыванием

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***Резюме.*** Цель – анализ устойчивости нейронных сетей Хопфилда с изменяющейся во времени задержкой. Для того чтобы система могла работать в устойчивом состоянии, важно гарантировать устойчивость нейронных сетей Хопфилда с изменяющейся во времени задержкой. Метод функционала Ляпунова-Красовского является основным методом исследования устойчивости систем с временной задержкой. На основе данного метода в работе анализируется устойчивость нейронных сетей Хопфилда с изменяющейся во времени задержкой. Известно, что из-за таких факторов, как время связи, ограниченная скорость переключения различных активных устройств, в различных технических системах часто возникают временные задержки, которые существенно ухудшают работу системы, что может в свою очередь приводить к полной потере устойчивости. В связи с этим в работе был построен функционал Ляпунова-Красовского типа «delay-product», что позволяет использовать больше информации о временной задержке и уменьшать консерватизм метода. Затем было использовано обобщенное интегральное неравенство на основе свободной матрицы. Сформулирован новый критерий асимптотической устойчивости нейронных сетей Хопфилда с изменяющейся во времени задержкой, который обладает меньшим консерватизмом. Проиллюстрирована эффективность предложенного метода. Таким образом, в работе сформулирован и обоснован критерий асимптотической устойчивости для нейронных сетей Хопфилда с изменяющейся во времени задержкой. При этом расширенный функционал Ляпунова-Красовского строится на основе запаздывания и квадратичного мультипликативного функционала, а производная функционала определяется матричным интегральным неравенством со свободными весами. Эффективность метода иллюстрируется на модельном примере.

***Ключевые слова:*** нейронные сети Хопфилда, асимптотическая устойчивость, метод функционала Ляпунова-Красовского

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## INTRODUCTION

It is well known that neural networks have many applications in the area of signal processing, pattern recognition etc. Hopfield neural network is one kind of neural networks given by J. J. Hopfield in 1982. In recent years, Hopfield neural networks (HNNs) has attracted an increasing attention since it has found many applications in classification of patterns, associative memory, image treatment, solving optimization problems and other areas [1–6]. Especially, HNNs have been widely applicated in power system. For example, HNNs can solve the economic dispatch, which is a typical optimal problem in power system, and give a proper dispatch bringing great economic benefits. What's more, in the electric power network planning, HNNs can be used to select each load node's in-degree and direction of in-degree and the structure of distribution can be also decided. The problems of power flow can be solved by HNNs in the same time. However, the time-varying delay usually exists in the HNNs and it has a negative influence on system performance. It means the existence of time-varying delay will make the performance of the system to be worse and even make it instable. Therefore, the stability analysis of HNNs is a hot topic.

In the existing literature, the stability analysis of HNNs is often based on time-invariant delays or based on simple Lyapunov-Krasovskii functional in [7–12]. However, the conclusions are conservative due to the less information about delays in the LKF. Therefore, this article will analyze the stability of HNNs with timevarying delays based on the augmented Lyapunov-Krasovskii functional with delay-product-type terms. [7] constructed a simple LKF for HNNs with time-invariant time delays, and obtained the stability criterion by using the time-delay segmentation method. The more time-delay segments, the lower the conservativeness, but the computational complexity also increased. [8] constructed an augmented LKF with more information about delays for HNNs with time-varying delays, and analyze its stability by the free weight matrix method. It still has room to decrease the conservativeness in terms of LKF construction and processing of functional derivatives. In terms of LKF construction, augmented LKF that contain more information of delays has been used widely. For example, [13] proposed two new LKFs with delay-product-type terms. The relationship between time delay and quadratic terms is changed from simple addition to multiplication. It can effectively reduce the conservatism of the conclusion. On the other hand, the derivative processing aspect of the functional is mainly changed from the free weight matrix method [14] to the integral inequality method, such as Jensen inequality [15], Wirtinger inequality [16], B-L inequality [17] , auxiliary function-based integral inequalities [18] and so on. The generalized free matrix integral inequality proposed in [19], which is based on Legendre series. By introducing some free matrices, the integral term can be estimated more tightly. Besides the two aspects mentioned above, how to find the condition that guarantees the negative definiteness of the derivative of LKFs is also important, especially when the derivative is a quadratic function with respect to the time-varying delay. The sufficient condition reported in [20] is commonly used but recently a relaxed quadratic function negative-determination lemma is proposed in [21]. In this lemma, an adjustable parameter is introduced which provides potential to reduce the conservatism without much computational complexity.

In this paper, a suitable LKF is constructed based on the delay and quadratic multiplication LKF, and the derivative of the LKF is estimated by the integral inequality method and a relaxed quadratic function negative-determination lemma is employed to obtain the asymptotic stability criteria of HNNs. Finally, a numerical example is given to demonstrate the advantages and effectiveness of the proposed method.

***Notation:*** The notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $P>0$ means that the matrix $P$ is positive definite; $I$ and $0$ represent an appropriately dimensioned identify matrix and zero matrix respectively; * stands for the symmetric term in the symmetric matrix; the transpose and the inverse of a matrix are denoted by the superscripts $T$ and −1; $Sym\{X\}=X+X^T$.

## PROBLEM FORMULATION

Consider the following Hopfield neural networks(HNNs):

$$\dot{y}(t)=-Ay(t)+Bg\left(y(t-\tau)\right)+u, \qquad (1)$$

wheredzDzdenotes the neuron state vector; $u=[u_1,u_2,...,u_n]^T\in\mathbb{R}^n$ is a constant input vector; $A=diag\{a_1,a_2,...,a_n\}>0$ is a diagonal matrix; $B$ is the delayed connection weight matrix;
$g(y(\bullet))=[g(y_1(\bullet)),g(y_2(\bullet)),...,g(y_n(\bullet))]^T\in\mathbb{R}^n$ is the neuron activation function; $\tau$ is the time delay.

***Assumption 1:*** The neuron activation function in system (1) for $\forall x,y\in\mathbb{R}$ and $x\neq y$ satisfies the following condition:

$$0\le\frac{g_i(x)-g_i(x)}{x-y}\le L_j,\ j=1,2,...,n, \qquad (2)$$

where $L_j(j=1,2,...,n)$ are positive constants.

Let $x^*=[x_1^*,x_2^*,...,x_n^*]^T$ be the equilibrium point of system (1). By the transformation $y(\bullet)=x(\bullet)-x^*$,we can simplify the equation and the HNNs (1) is rewritten as:

$$\dot{x}(t)=-Ax(t)+Bf\left(x(t-\tau)\right), \qquad (3)$$

where

$$f_i(x_i(\bullet))=g_i(x_i(\bullet)+x_i^*)-g_i(x_i^*), i=1,2,...,n. \qquad (4)$$

By Assumption (1) and (4), it is easy to verify that

$$0\le\frac{f_j(x_j)}{x_j}\le L_j,\ f_j(0)= =0, \forall x_j\in\mathbb{R},\ j=1,2,...,n. \qquad (5)$$

The time-varying delay satisfies the following condition:

$$0<\tau<h, 0<\dot{\tau}<\mu. \qquad (5)$$

***Lemma 1:*** Given a positive integer [19] $N$, an positive definite symmetric matrix $R\in\mathbb{R}^n$, $M_i\in\mathbb{R}^{n\times m}(i=0,1,2)$ and a vector $\zeta\in\mathbb{R}^m$, for any continuous differentiable function, the following inequality holds:

$$\int_\alpha^\beta \dot{x}^T(s)R\dot{x}(s)ds\le \le\sum_{k=0}^{N}[2\zeta_N^T\pi_N^T M_k\zeta+\frac{\beta-\alpha}{2k+1}\zeta_N^T M_k^T R^{-1}M_k\zeta],$$

where

$$\zeta_N=\begin{cases}[x^T(\beta)\ \ x^T(\alpha)]^T, & N=0\\ [x^T(\beta)\ \ x^T(\alpha)\ \ \frac{1}{\beta-\alpha}\Omega_0^T\ \cdots\ \frac{1}{\beta-\alpha}\Omega_{N-1}^T]^T, & N>0\end{cases}$$

$$\pi_N(k)=\begin{cases}[I\ \ -I], & N=0\\ [I\ \ (-1)^{k+1}I\ \ \lambda_{Nk}^0 I\cdots\lambda_{Nk}^{N-1}I], & N>0\end{cases}$$

$$\Omega_k=\int_\alpha^\beta L_k(u)x(u)du$$

$$\lambda_{Nk}^{i} = \begin{cases} -(2i+1)(1-(-1)^{k+i}), & i \le k \\ 0, & i \ge k+1 \end{cases}$$

$$L_k(s) = (-1)^k \sum_{l=0}^{k} \left[ (-1)^l \binom{k}{l} \binom{k+l}{l} \right] \left( \frac{s-\alpha}{\beta-\alpha} \right)^l .$$

***Lemma 2:*** For a quadratic function [21] $f(y) = a_2 y^2 + a_1 y + a_0$, $f(y) < 0$ holds for $y \in [\alpha, \beta]$ if the following holds for any given $\kappa \in [0,1]$:

$$\begin{cases} f(\alpha) < 0 \\ f(\beta) < 0 \\ -\kappa^2 a_2 (\beta-\alpha)^2 + f(\alpha) < 0 \\ -(1-\kappa)^2 a_2 (\beta-\alpha)^2 + f(\beta) < 0 \end{cases} .$$

***Lemma 3:*** Assuming that [22] (5) holds, then

$$\int_{\alpha}^{\beta} \left[ f_i(s) - f_i(s) \right] ds \le \left[ \beta - \alpha \right] \left[ f_i(\beta) - f_i(\alpha) \right], i = 1, 2, ..., n.$$

## ASYMPTOTIC STABILITY CRITERIA

In this section, a new delay-dependent asymptotic stability criterion for HNNs with time-varying delay is derived.

***Theorem 1:*** Given a fixed $\kappa \in [0,1]$, $h$, $\mu_1$, $\mu_2$, system (3) is asymptotically stable if there exist symmetric matrix $P_i > 0 (i = 1,2,3)$, $Q_i > 0 (i = 1,2,3,4)$, $R > 0$, matrix $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\} > 0$, $S_i = diag\{s_{i1}, s_{i2}, ..., s_{in}\} > 0$ $(i = 1,2)$ and any matrix $M_{1i}, M_{2i}$ $(i = 0,1,2)$, such that the following holds

$$\begin{bmatrix} \Upsilon(h_l, \mu_c) & hM_1^T \\ * & -h\hat{R} \end{bmatrix} < 0; \quad (6)$$

$$\begin{bmatrix} \Upsilon(h_l, \mu_c) + \Upsilon(\kappa h, \mu_c) - \delta_l^2 a_2 h^2 & h\tilde{M}_l^T \\ * & -h\tilde{R}_l \end{bmatrix} < 0, \quad (7)$$

where $l = 1,2$; $c = 1,2$; $h_1 = 0$; $h_2 = h$; $\mu_1 = 0$; $\mu_2 = \mu$, $\delta_1 = \kappa$; $\delta_2 = 1-\kappa$; $\bar{\tau} = h - \tau$;

$$M_l = col\{M_{l0}, M_{l1}, M_{l2}\};$$

$$\tilde{M}_l = col\{M_l, (1-\kappa)M_1, \kappa M_2\};$$

$$\tilde{R}_l = diag\{\hat{R}, (1-\kappa)\hat{R}, \kappa\hat{R}\},$$

$$\hat{R} = diag\{R, 3R, 5R\};$$

$$e_i = [0_{n\times(i-1)n}\ , I\ , 0_{n\times(12-i)n}], i = 1, 2, ..., 12;$$

$$e_s = -Ae_1 + Be_{11};$$

$$\Upsilon(\tau, \dot{\tau}) = \Upsilon_1(\tau, \dot{\tau}) + \Upsilon_2(\dot{\tau}) + \Upsilon_3(\dot{\tau}) + \sum_{i=4}^{6} \Upsilon_i;$$

$$\Upsilon_1(\tau) = Sym\left\{\Pi_{11}^T P_1 \Pi_{21}\right\} + \dot{\tau}\,\Pi_{12}^T P_2 \Pi_{12} + \tau Sym\left\{\Pi_{12}^T P_2 \Pi_{22}\right\} - \dot{\tau}\,\Pi_{13}^T P_3 \Pi_{13} + \bar{\tau} Sym\left\{\Pi_{13}^T P_3 \Pi_{23}\right\};$$

$$\Upsilon_2(\dot{\tau}) = e_1^T Q_1 e_1 - (1-\dot{\tau}) e_2^T Q_1 e_2 + (1-\dot{\tau}) e_2^T Q_2 e_2 - e_3^T Q_2 e_3;$$

$$\Upsilon_3(\dot{\tau}) = \Pi_3^T Q_3 \Pi_3 - (1-\dot{\tau}) \Pi_4^T Q_3 \Pi_4 + (1-\dot{\tau}) \Pi_4^T Q_4 \Pi_4 - \Pi_5^T Q_4 \Pi_5;$$

$$\Upsilon_4 = Sym\left\{e_{10}^T \Lambda e_s\right\};$$

$$\Upsilon_5 = he_s^T R\, \mathrm{e}_s + \sum_{k=0}^{2} Sym\left\{g_1^T E_{1k}^T M_{1k} + g_2^T E_{1k}^T M_{2k}\right\};$$

$$\Upsilon_6 = Sym\{e_{10}^T \Lambda e_s + e_1^T L S_1 e_{10} - e_{10}^T S_1 e_{10} + e_2^T L S_2 e_{11} - e_{11}^T S_2 e_{11}\};$$

$$a_2 = \dot{\tau} N_1^T P_2 N_1 + Sym\left\{N_1^T P_2 \Pi_{22}\right\} - \dot{\tau} N_2^T P_3 N_2 + Sym\left\{N_2^T P_3 \Pi_{23}\right\};$$

$$d_1 = col\{e_1, e_2, e_3\};$$

$$d_2 = col\{e_4, e_5\}, \ d_3 = col\{e_6, e_7\};$$

$$d_4 = col\{e_s, (1-\dot{\tau})e_8, e_9\};$$

$$d_5 = col\{e_1 - (1-\dot{\tau})e_2, e_1 - (1-\dot{\tau})e_4 - \dot{\tau}e_5\};$$

$$d_6 = col\{(1-\dot{\tau})e_2 - e_3, (1-\dot{\tau})e_2 - e_6 + \dot{\tau}e_7\};$$

$$\Pi_{11} = col\{d_1, \tau d_2, \bar{\tau} d_3\}, \ \Pi_{21} = col\{d_4, d_5, d_6\};$$

$$\Pi_{12} = col\{d_1, \tau d_2\}, \ \Pi_{22} = col\{d_4, d_5\};$$

$$\Pi_{13} = col\{d_1, \bar{\tau} d_3\}, \ \Pi_{23} = col\{d_4, d_6\};$$

$$\Pi_3 = col\{e_s, e_1, e_{10}\}, \ \Pi_4 = col\{e_8, e_2, e_{11}\};$$

$$\Pi_5 = col\{e_9, e_3, e_{12}\}, \ E_{10} = [I, -I, 0, 0];$$

$$E_{11} = [I, I, -2I, 0], \ E_{12} = [I, -I, 0, -6I];$$

$$g_1 = col\{e_2, e_3, e_6, -e_6 + 2e_7\};$$

$$g_2 = col\{e_1, e_2, e_4, -e_4 + 2e_5\};$$

$$N_1 = col\{0,0,0,d_2\}, \ N_2 = col\{0,0,0,d_3\};$$

$$\xi(t) = col\{x(t), x(t-\tau), x(t-h), \int_{t-\tau}^{t}\frac{x(s)}{\tau}ds, \int_{t-\tau}^{t}\int_{\theta}^{t}\frac{x(s)}{\tau^2}dsd\theta, \int_{t-h}^{t-\tau}\frac{x(s)}{\bar{\tau}}ds, \int_{t-h}^{t-\tau}\int_{\theta}^{t-\tau}\frac{x(s)}{\bar{\tau}^2}dsd\theta, \dot{x}(t-\tau), \dot{x}(t-h), f(x(t)), f(x(t-\tau)), f(x(t-h))\}.$$

***Proof:*** Consider the following LKF candidate:

$$V(t) = \sum_{i=1}^{5} V_i(t), \quad (8)$$

where

$$V_1(t) = \eta_1^T(t)P_1\eta_1(t) + \tau\eta_2^T(t)P_2\eta_2(t) + (h-\tau)\eta_3^T(t)P_3\eta_3(t);$$

$$V_2(t) = \int_{t-\tau}^{t} x^T(s)Q_1x(s)ds;$$

$$V_3(t) = \int_{t-\tau}^{t} \eta_2^T(s)Q_3\eta_2(s)ds;$$

$$V_4(t) = 2\sum_{i=1}^{n}\lambda_i\int_0^{x_j} f_i(s)ds;$$

$$V_5(t) = \int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)R\dot{x}(s)dsd\theta;$$

and

$$\eta_1(t) = col\{x(t), x(t-\tau), x(t-h), \int_{t-\tau}^{t}\frac{x(s)}{\tau}ds, \int_{t-\tau}^{t}\int_{\theta}^{t}\frac{x(s)}{\tau^2}dsd\theta, \int_{t-h}^{t-\tau}\frac{x(s)}{\bar{\tau}}ds, \int_{t-h}^{t-\tau}\int_{\theta}^{t-\tau}\frac{x(s)}{\bar{\tau}^2}dsd\theta\};$$

$$\eta_2(t) = col\{x(t), x(t-\tau), x(t-h), \int_{t-\tau}^{t}\frac{x(s)}{\tau}ds, \int_{t-\tau}^{t}\int_{\theta}^{t}\frac{x(s)}{\tau^2}dsd\theta\};$$

$$\eta_3(t) = col\{x(t), x(t-\tau), x(t-h), \int_{t-h}^{t-\tau}\frac{x(s)}{\bar{\tau}}ds, \int_{t-h}^{t-\tau}\int_{\theta}^{t-\tau}\frac{x(s)}{\bar{\tau}^2}dsd\theta\}$$

and $P_i > 0 (i = 1,2,3)$, $Q_i > 0 (i = 1,2,3,4)$, $R > 0$, $\Lambda > 0$ which shows $V(t) \geq \varepsilon\|x\|^2$ for a sufficient small $\varepsilon > 0$.

Then calculating the derivatives of $V(t)$ defined in (9) and the derivatives of $V_i(t)$ $(i = 1,2,...,5)$ are given by

$$\dot{V}_1(t) = 2\eta_1^T(t)P_1\dot{\eta}_1(t) + \dot{\tau}\eta_2^T(t)P_2\eta_2(t) + 2\tau\eta_2^T(t)P_2\dot{\eta}_2(t) - \dot{\tau}\eta_3^T(t)P_3\eta_3(t) + 2\bar{\tau}\eta_3^T(t)P_3\dot{\eta}_3(t) = \xi^T(t)\Upsilon_1(\tau,\dot{\tau})\xi(t). \quad (9)$$

$$\dot{V}_2(t) = x^T(t)Q_1x(s) - x^T(t-\tau)(Q_1 - Q_2) x(t-\tau) - Q_2x(t-h) = \xi^T(t)\{e_1^TQ_1e_1 - e_2^TQ_1e_2 + e_2^TQ_2e_2 - e_3^TQ_2e_3\}\xi(t) = \xi^T(t)\Upsilon_2(\dot{\tau})\xi(t). \quad (10)$$

$$\begin{aligned}\dot{V}_3(t) &= \eta_2^T(t)Q_3\eta_2(t) - \eta_2^T(t-\tau)(Q_3-Q_4)\\ &\quad \eta_2(t-\tau) - \eta_2^T(t-h)Q_4\eta_2(t-h)\\ &= \xi^T(t)\{\Pi_3^T Q_3 \Pi_3 - \Pi_4^T Q_3 \Pi_4\\ &\quad + \Pi_4^T Q_4 \Pi_4 - \Pi_5^T Q_4 \Pi_5\}\xi(t)\\ &= \xi^T(t)\Upsilon_3(\dot{\tau})\xi(t).\end{aligned} \quad (11)$$

$$\begin{aligned}\dot{V}_4(t) &= 2f^T(x(t))\Lambda\dot{x}(t)\\ &= \xi^T(t)Sym\left\{e_{10}^T\Lambda e_s\right\}\xi(t)\\ &= \xi^T(t)\Upsilon_4\xi(t).\end{aligned} \quad (12)$$

$$\dot{V}_5(t) = h\dot{x}^T(t)R\dot{x}(t) - \int_{t-h}^{t}\dot{x}^T(t)R\dot{x}(t)ds. \quad (13)$$

Based on lemma 1, $\int_{t-h}^{t}\dot{x}^T(t)R\dot{x}(t)ds$ with $R>0$ can be estimated as:

$$\begin{aligned}&-\int_{t-h}^{t}\dot{x}^T(t)R\dot{x}(t)ds = -\int_{t-h}^{t-\tau}\dot{x}^T(t)R\dot{x}(t)ds -\\ &-\int_{t-\tau}^{t}\dot{x}^T(t)R\dot{x}(t)ds \le \sum_{k=0}^{2}[2\zeta_1^T(t)E_{1k}^T M_{1k}\xi(t)\\ &\quad + \frac{\bar{\tau}}{2k+1}\xi^T(t)M_{1k}^T R^{-1}\tilde{M}_{1k}\xi(t)]\\ &\quad + \sum_{k=0}^{2}[2\zeta_2^T(t)E_{1k}^T M_{2k}\xi(t)\\ &\quad + \frac{\bar{\tau}}{2k+1}\xi^T(t)M_{2k}^T R^{-1}\tilde{M}_{2k}\xi(t)]\\ &= \xi^T(t)\left\{\sum_{k=0}^{2}Sym\left\{g_1^T E_{1k}^T M_{1k} + g_2^T E_{1k}^T M_{2k}\right\} + \bar{\Upsilon}_5\right\}\xi(t).\end{aligned}$$

where

$$\begin{aligned}\zeta_1(t) &= col\{x(t-\tau), x(t-h), \frac{1}{\bar{\tau}}\int_{t-h}^{t-\tau}x(s)ds,\\ &\quad -\frac{1}{\bar{\tau}}\int_{t-h}^{t-\tau}x(s)ds + \frac{2}{\bar{\tau}^2}\int_{t-h}^{t-\tau}\int_{s}^{t-\tau}x(s)dsdu\};\end{aligned}$$

$$\begin{aligned}\zeta_2(t) &= col\{x(t), x(t-\tau), \frac{1}{\tau}\int_{t-\tau}^{t}x(s)ds,\\ &\quad -\frac{1}{\tau}\int_{t-\tau}^{t}x(s)ds + \frac{2}{\tau^2}\int_{t-\tau}^{t}\int_{s}^{t}x(u)duds\}.\end{aligned}$$

Therefore,

$$\dot{V}_5(t) \le \xi^T(t)\left\{\Upsilon_5 + \bar{\Upsilon}_5(\tau)\right\}\xi(t). \quad (14)$$

For matrices $S_i = diag\left\{s_{i1}, s_{i2}, ..., s_{in}\right\} > 0$ $(i=1,2)$, the following hold by using lemma 3:

$$\begin{aligned}&0 \le 2x^T(t)LS_1 f\left(x(t)\right) - 2f^T\left(x(t)\right)S_1 f\left(x(t)\right)\\ &\quad + 2x^T(t-\tau)LS_2 f\left(x(t-\tau)\right)\\ &\quad -2f^T\left(x(t-\tau)\right)S_2 f\left(x(t-\tau)\right)\\ &\quad = \xi^T(t)\Upsilon_6\xi(t).\end{aligned} \quad (15)$$

It follows from (10)–(16) that

$$\dot{V}\left(t\right) \le \xi^T(t)\left(\Upsilon(\tau) + \bar{\Upsilon}_5\right)\xi(t) = \varpi(\tau). \quad (16)$$

It is found that $\varpi(\tau,\dot{\tau})$ is a quadratic function. Thus, based on lemma 2, the following holds:

$$\begin{cases}\varpi(0,0) < 0\\ \varpi(h,0) < 0\\ -\kappa^2 a_2 h^2 + \varpi(0,0) < 0\\ -(1-\kappa)^2 a_2 h^2 + \varpi(h,0) < 0\end{cases}; \quad (17)$$

$$\begin{cases}\varpi(0,\mu) < 0\\ \varpi(h,\mu) < 0\\ -\kappa^2 a_2 h^2 + \varpi(0,\mu) < 0\\ -(1-\kappa)^2 a_2 h^2 + \varpi(h,\mu) < 0\end{cases}. \quad (18)$$

It follows that $(18),(19) \Rightarrow (7),(8)$ by using Schur complement. This completes the proof.

**NUMERICAL EXAMPLES**

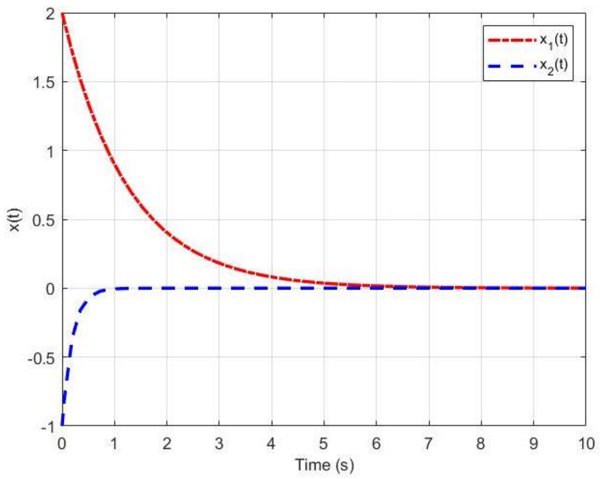
In this section, a numerical example is given to demonstrate the effectiveness and advantages of the proposed method.

***Example 1:*** Consider the following HNN (3) with time-varying delay (6):

$$A = \begin{bmatrix}0.8 & 0\\ 0 & 5.3\end{bmatrix}, \quad B = \begin{bmatrix}0.1 & 0.3\\ 0.9 & 0.1\end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mu = 0.9.$$

By the Theorem 1 of this paper, the maximum delay h that guarantees the asymptotic stability of HNN (3) is 51.3012 while it is 14.660 in [8]. So our result is less conservative.




***State trajectories of the system of Example 1***
***Траектории состояний системы примера 1***

Setting

$$x(0) = col\{2, -1\} \ \tau = 0.9\sin t + 50.4012$$
$$f(x) = col\{\tanh(x_1(t)), \tanh(x_2(t))\}.$$

The responses of the HNN (3) with a time-varying delay when $\mu = 0.9$, $h = 51.3012$ are shown in the figure above. The result indicates that the system is stable at its equilibrium points, which verifies the effectiveness of the proposed method.

## CONCLUSION

In this paper, an augmented functional is constructed based on the delay and quadratic multiplicative functional, and the derivative of the functional is defined by the free-weight matrix integral inequality. We choose the relaxed quadratic function negative-determination lemma to deal with the quadratic function and obtain the stability criterion. Finally, a numerical example is given to prove the effectiveness and advantages of the proposed method in this paper.

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